

Combining Models of Approximation with Partial Learning^{*}

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Abstract. In Gold’s framework of inductive inference, the model of partial learning requires the learner to output exactly one correct index for the target object and only the target object infinitely often. Since infinitely many of the learner’s hypotheses may be incorrect, it is not obvious whether a partial learner can be modified to “approximate” the target object.

Fulk and Jain (Approximate inference and scientific method. *Information and Computation* 114(2):179–191, 1994) introduced a model of approximate learning of recursive functions. The present work extends their research and solves an open problem of Fulk and Jain by showing that there is a learner which approximates and partially identifies every recursive function by outputting a sequence of hypotheses which, in addition, are also almost all finite variants of the target function.

The subsequent study is dedicated to the question how these findings generalise to the learning of r.e. languages from positive data. Here three variants of approximate learning will be introduced and investigated with respect to the question whether they can be combined with partial learning. Following the line of Fulk and Jain’s research, further investigations provide conditions under which partial language learners can eventually output only finite variants of the target language.

1 Introduction

Gold [10] considered a learning scenario where the learner is fed with piecewise increasing amounts of finite data about a given target language L ; at every stage where a new input datum is given, the learner makes a conjecture about L . If

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there is exactly one correct representation of L that the learner always outputs after some finite time (assuming that it never stops receiving data about L), then the learner is said to have “identified L in the limit.” In this paper, it is assumed that all target languages are encoded as recursively enumerable (r.e.) sets of natural numbers, and that the learner uses Gödel numbers as its hypotheses.

Gold’s learning paradigm has been used as a basis for a variety of theoretical models in subjects such as human language acquisition [18] and the theory of scientific inquiry in the philosophy of science [4,17]. This paper is mainly concerned with the *partial learning* model [19], which retains several features of Gold’s original framework – the modelling of learners as recursive functions, the use of texts as the mode of data presentation and the restriction of target classes to the family of all r.e. sets – while liberalising the learning criterion by only requiring the learner to output exactly one hypothesis of the target set infinitely often while it must output any other hypothesis only finitely often. It is known that partial learning is so powerful that the class of all r.e. languages can be partially learnt [19].

However, the model of partial learning puts no further constraints on those hypotheses that are output only finitely often. In particular, it offers no notion of “eventually being correct” or even “approximating” the target object. From a philosophical point of view, if partial learning is to be taken seriously as a model of language acquisition, then it is quite plausible that learners are capable of gradually improving the quality of their hypotheses over time. For instance, if the learner M sees a sentence S in the text at some point, then it is conceivable that after some finite time, M will only conjecture grammars that generate S . This leads one to consider a notion of the learner “approximating” the target language.

The central question in this paper is whether any partial learner can be redefined in a way that it approximates the target object and still partially learns it. The first results, in the context of partial learning, deal with Fulk and Jain’s [5] notion of approximating recursive functions. Fulk and Jain proved the existence of a learner that “approximates” every recursive function. This result is generalised as follows: partial learners can always be made to approximate recursive functions according to their model and, in addition, eventually output only finite variants of the target function, that is, they can be designed as BC^* learners⁴. This result solves an open question posed by Fulk and Jain, namely whether recursive functions can be approximated by BC^* learners. Note that BC^* learning can also, in some sense, be considered a form of approximation, as it requires that eventually all of the hypotheses (including those output only finitely often) differ from the target object in only finitely many values. It thus is interesting to see that partial learning can be combined not only with Fulk and Jain’s model of approximation, but also with BC^* learning *at the same time*. Note that in this paper, when two learning criteria A and B are said to be combinable, it is generally not assumed that the new learner is effectively constructed from the A -learner and the B -learner.

⁴ BC^* is mnemonic for “behaviourally correct with finitely many anomalies” [4].

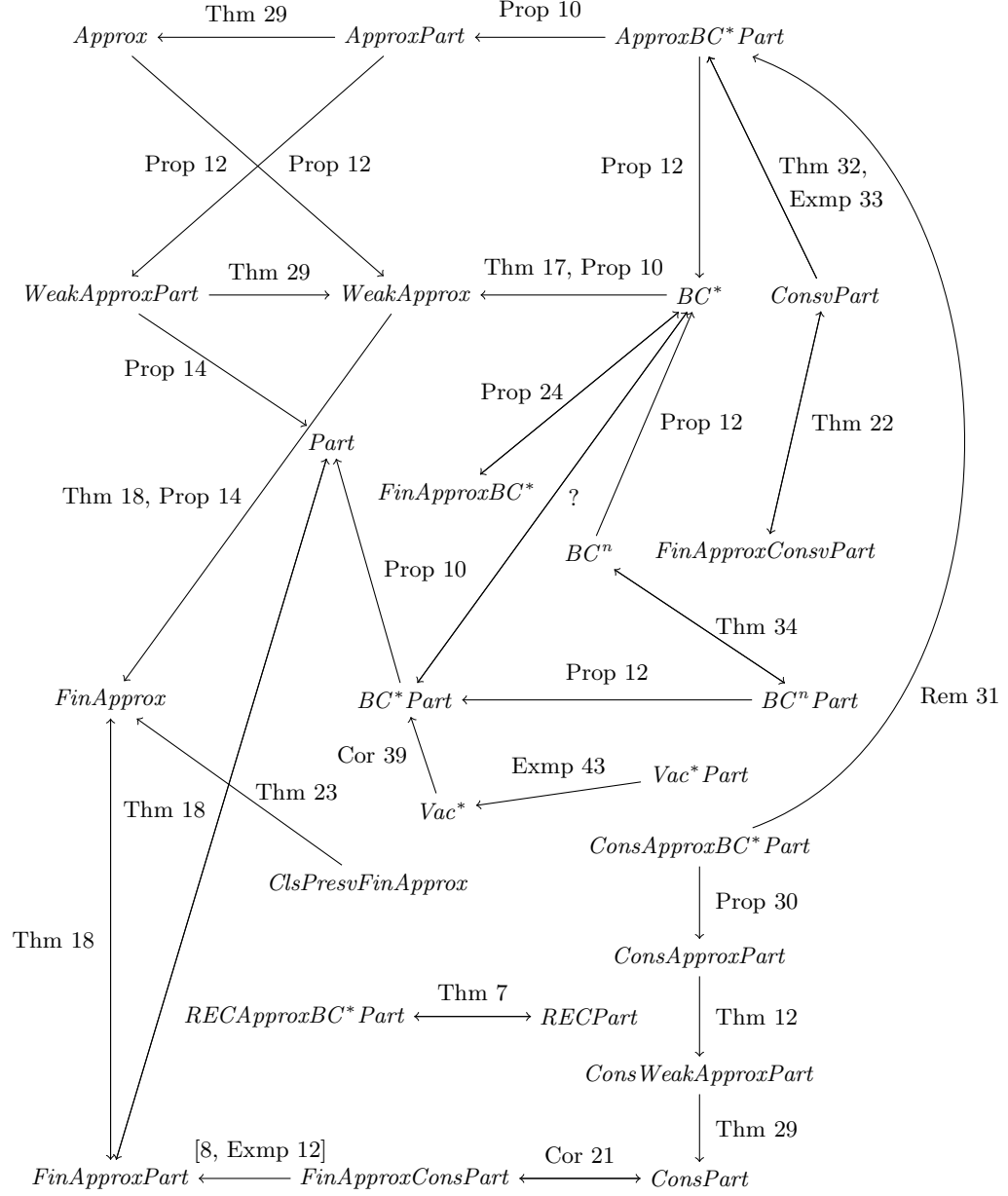
This raises the question whether partial learners can also be turned into approximate learners in the more general case of learning r.e. languages. Unfortunately, Fulk and Jain’s model applies only to learning recursive functions. The second contribution is the design of three notions of approximate learning of r.e. languages, two of which are directly inspired by Fulk and Jain’s model. It is then investigated under which conditions partial learners can be modified to fulfill the corresponding constraints of approximate learning. These investigations are also extended to partial learners with additional constraints, such as consistency and conservativeness. It will be shown that while partial learners can always be constructed in a way so that for any given finite set D , their hypotheses will almost always agree with the target language on D , the same does not hold if D must be a finite variant of a fixed infinite set. Thus trade-offs between certain approximate learning constraints and partial learning are sometimes unavoidable – an observation that perhaps has a broader implication in the philosophy of language learning.

Following the line of Fulk and Jain’s research, conditions are investigated under which partial language learners can eventually output only finite variants of the target function. While it remains open whether or not partial learners for a given BC^* -learnable class can be made BC^* -learners for this class without losing identification power, some natural conditions on a BC^* learner M are provided under which all classes learnable by M can be learnt by some BC^* learner N that is at the same time a partial learner.

Figure 1 summarises the main results of this paper. *RECPart* and *RECAproxBC*Part* refer respectively to partial learning of recursive functions and approximate BC^* partial learning of recursive functions. The remaining learning criteria are abbreviated (see Definitions 3, 4 and 8), and denote learning of classes of r.e. languages. An arrow from criterion A to criterion B means that the collection of classes learnable under model A is contained in that learnable under model B . Each arrow is labelled with the Corollary/Example/Remark/Theorem number(s) that proves (prove) the relationship represented by the arrow. If there is no path from A to B , then the collection of classes learnable under model A is not contained in that learnable under model B .

2 Preliminaries

The notation and terminology from recursion theory adopted in this paper follows in general the book of Rogers [20]. Background on inductive inference can be found in [11]. The symbol \mathbb{N} denotes the set of natural numbers, $\{0, 1, 2, \dots\}$. Let $\varphi_0, \varphi_1, \varphi_2, \dots$ denote a fixed acceptable numbering [20] of all partial-recursive functions over \mathbb{N} . Given a set S , S^* denotes the set of all finite sequences in S . Wherever no confusion may arise, S will also denote its own characteristic function, that is, for all $x \in \mathbb{N}$, $S(x) = 1$ if $x \in S$ and $S(x) = 0$ otherwise. One defines the e -th r.e. set W_e as $\text{dom}(\varphi_e)$ and the e -th canonical finite set by choosing D_e such that $\sum_{x \in D_e} 2^x = e$. This paper fixes a one-one padding function pad with $W_{\text{pad}(e,d)} = W_e$ for all e, d . Furthermore, $\langle x, y \rangle$ denotes Cantor’s

**Fig. 1.** Learning hierarchy

pairing function, given by $\langle x, y \rangle = \frac{1}{2}(x + y)(x + y + 1) + y$. A triple $\langle x, y, z \rangle$ denotes $\langle \langle x, y \rangle, z \rangle$. The notation $\eta(x) \downarrow$ means that $\eta(x)$ is defined, and $\eta(x) \uparrow$ means that $\eta(x)$ is undefined. The notation $\varphi_e(x) \uparrow$ means that $\varphi_e(x)$ remains

undefined and $\varphi_{e,s}(x) \downarrow$ means that $\varphi_e(x)$ is defined within s steps, that is, the computation of $\varphi_e(x)$ halts within s steps. K denotes the halting problem, that is, $K = \{x : \varphi_x(x) \downarrow\}$. For any r.e. set A , A_s denotes the s th approximation of A ; it is assumed that for all s , $|A_{s+1} - A_s| \leq 1$ and $A_s \subseteq \{0, \dots, s\}$.

For any $\sigma, \tau \in (\mathbb{N} \cup \{\#\})^*$, $\sigma \preceq \tau$ if and only if σ is a prefix of τ , $\sigma \prec \tau$ if and only if σ is a proper prefix of τ , and $\sigma(n)$ denotes the element in the n th position of σ , starting from $n = 0$. For each $\sigma \neq \epsilon$, σ' denotes the string obtained from σ by deleting the last symbol of σ . The concatenation of two strings σ and τ shall be denoted by $\sigma \circ \tau$; for convenience, and whenever there is no possibility of confusion, this is occasionally denoted by $\sigma\tau$. Let $\sigma[n]$ denote the sequence $\sigma(0) \circ \sigma(1) \circ \dots \circ \sigma(n-1)$. The length of σ is denoted by $|\sigma|$.

3 Learning

The basic learning paradigms studied in the present paper are *behaviourally correct learning* [2,3] and *partial learning* [19]. These learning models assume that the learner is presented with just positive examples of the target language, and that the learner is fed with a finite amount of data at every stage. They are modifications of the model of explanatory learning (or “learning in the limit”), first introduced by Gold [10], in which the learner must output in the limit a single correct representation h of the target language L ; if L is an r.e. set, then h is usually an r.e. index of L with respect to the standard numbering W_0, W_1, W_2, \dots of all r.e. sets. Bärzdīņš [2] and Case [3] considered the more powerful model of behaviourally correct learning, whereby the learner must almost always output a correct hypothesis of the input set, but some of the correct hypotheses may be syntactically distinct. Case and Smith [4] also introduced a less stringent variant of *BC* learning of recursive functions, *BC** learning, which only requires the learner to output in the limit finite variants of the target recursive function. Still more general is the criterion of partial learning that Osherson, Stob and Weinstein [19] defined; in this model, the learner must output exactly one correct index of the input set infinitely often and output any other conjecture only finitely often.

One can also impose constraints on the quality of a learner’s hypotheses. For example, Angluin [1] introduced the notion of *consistency*, which is the requirement that the learner’s hypotheses must enumerate at least all the data seen up to the current stage. This seems to be a fairly natural demand on the learner, for it only requires that the learner’s conjectures never contradict the available data on the target language. Angluin [1] also introduced the learning constraint of *conservativeness*; intuitively, a conservative learner never makes a mind change unless its prior conjecture does not enumerate all the current data. A further constraint proposed by Osherson, Stob and Weinstein [18] is *confidence*, according to which the learner must converge on any (even non r.e.) text. These three learning criteria have since been adapted to the partial learning model [7,8].

Lange and Zeugmann [15] showed that learning in the limit is less powerful if the hypothesis space of the learner is restricted to the target class. It would thus be quite natural to ask whether this constraint on the hypothesis space of the learner has a similar effect on partial learning or on approximate learning. For this purpose, one distinguishes between *class-comprising* learning and *class-preserving* learning [15]. If the learner M only conjectures languages that it can successfully learn, then M is said to be *prudent* [18]. The learning criteria discussed so far (and, where applicable, their partial learning analogues) are formally introduced below.

Definition 1. [21] M is said to *class-comprisingly* learn \mathcal{C} if it learns \mathcal{C} with respect to a hypothesis space $\{H_0, H_1, H_2, \dots\}$, where H_0, H_1, H_2, \dots are r.e. sets, such that $\mathcal{C} \subseteq \{H_0, H_1, H_2, \dots\}$.

Definition 2. [21] M is said to *class-preservingly* (*ClsPresv*) learn \mathcal{C} if it learns \mathcal{C} with respect to a hypothesis space $\{H_0, H_1, H_2, \dots\}$, where H_0, H_1, H_2, \dots are r.e. sets, such that $\mathcal{C} = \{H_0, H_1, H_2, \dots\}$.

Throughout this paper, successful learning with respect to a class \mathcal{C} will generally refer to class-comprising learning unless specified otherwise.

The learning criteria discussed so far (and, where applicable, their partial learning analogues) are formally introduced below.

Let \mathcal{C} be a class of r.e. sets. Throughout this paper, the mode of data presentation is that of a *text*, by which is meant an infinite sequence of natural numbers and the $\#$ symbol. Formally, a *text* T_L for some L in \mathcal{C} is a map $T_L : \mathbb{N} \rightarrow \mathbb{N} \cup \{\#\}$ such that $L = \text{range}(T_L)$; here, $T_L[n]$ denotes the sequence $T_L(0) \circ T_L(1) \circ \dots \circ T_L(n-1)$ and the range of a text T , denoted $\text{range}(T)$, is the set of numbers occurring in T . Analogously, for a finite sequence σ , $\text{range}(\sigma)$ is the set of numbers occurring in σ . A text, in other words, is a presentation of positive data from the target set. A *learner*, denoted by M in the following definitions, is a recursive function mapping $(\mathbb{N} \cup \{\#\})^*$ into \mathbb{N} . M may also be equipped with an oracle. In this case, a learner that has access to oracle A is an A -recursive function mapping $(\mathbb{N} \cup \{\#\})^*$ into \mathbb{N} .

- Definition 3.** (I) [19] M *partially* (*Part*) learns \mathcal{C} if, for every L in \mathcal{C} and each text T_L for L , there is exactly one index e such that $M(T_L[k]) = e$ for infinitely many k ; furthermore, if M outputs e infinitely often on T_L , then $L = W_e$.
- (II) [3] M *behaviourally correctly* (*BC*) learns \mathcal{C} if, for every L in \mathcal{C} and each text T_L for L , there is a number n for which $L = W_{M(T_L[j])}$ whenever $j \geq n$.
- (III) [1] M is *consistent* (*Cons*) if for all $\sigma \in (\mathbb{N} \cup \{\#\})^*$, $\text{range}(\sigma) \subseteq W_{M(\sigma)}$.
- (IV) [1] For any text T , M is *consistent on* T if $\text{range}(T[n]) \subseteq W_{M(T[n])}$ for all $n > 0$.
- (V) [8] M is said to *consistently partially* (*ConsPart*) learn \mathcal{C} if it partially learns \mathcal{C} from text and is consistent.

- (VI) [7] M is said to *conservatively partially* (*ConsvPart*) learn \mathcal{C} if it partially learns \mathcal{C} and outputs on each text for every L in \mathcal{C} exactly one index e with $L \subseteq W_e$.
- (VII) [8] M is said to *confidently partially* (*ConfPart*) learn \mathcal{C} if it partially learns \mathcal{C} from text and outputs on every infinite sequence (including sequences that are not texts for any member of \mathcal{C}) exactly one index infinitely often.
- (VIII) [4] M is said to *behaviourally correctly* learn \mathcal{C} *with at most a anomalies* (BC^a) iff for every $L \in \mathcal{C}$ and each text T_L for L , there is a number n for which $|(W_{M(T_L[j])} - L) \cup (L - W_{M(T_L[j])})| \leq a$ whenever $j \geq n$.
- (IX) [4] M is said to *behaviourally correctly* learn \mathcal{C} *with finitely many anomalies* (BC^*) iff for every $L \in \mathcal{C}$ and each text T_L for L , there is a number n for which $|(W_{M(T_L[j])} - L) \cup (L - W_{M(T_L[j])})| < \infty$ whenever $j \geq n$.

This paper will also consider combinations of different learning criteria; for learning criteria A_1, \dots, A_n , a class \mathcal{C} is said to be $A_1 \dots A_n$ -learnable iff there is a learner M such that M A_i -learns \mathcal{C} for all $i \in \{1, \dots, n\}$.

4 Approximate Learning of Functions

Fulk and Jain [5] proposed a mathematically rigorous definition of *approximate inference*, a notion originally motivated by studies in the philosophy of science.

Definition 4. [5] An approximate (*Approx*) learner outputs on the graph of a function f a sequence of hypotheses such that there is a sequence S_0, S_1, \dots of sets satisfying the following conditions:

- (a) The S_n form an ascending sequence of sets such that their union is the set of all natural numbers;
- (b) There are infinitely many n such that $S_{n+1} - S_n$ is infinite;
- (c) The n -th hypothesis is correct on all $x \in S_n$ but nothing is said about the $x \notin S_n$.

The next proposition simplifies this set of conditions.

Proposition 5. M *Approx* learns a recursive function f iff the following conditions hold:

- (d) For all x and almost all n , M 's n -th hypothesis is correct at x ;
- (e) There is an infinite set S such that for almost all n and all $x \in S$, M 's n -th hypothesis is correct at x .

Proof. If one has (a), (b), (c), then the set S is just the first set S_n which is infinite and the other conditions follow.

If one has (d) and (e) and one distinguishes two cases: If n is so small that the n -th and all subsequent hypotheses are not yet correct on S then one lets $S_n = \emptyset$ else one defines that S_n contains all $x \leq n$ such that each m -th hypothesis with $m \geq n$ is correct on x plus half of those members of S which are not in any S_m with $m < n$. So the trick is just not to put all members of S at one step into

some S_n but just to put at each step which is applicable an infinite new amount while still another infinite amount remains outside S_n to be put later. ■

Fulk and Jain interpreted their notion of approximation as a process in scientific inference whereby physicists take the limit of the average result of a sequence of experiments. Their result that the class of recursive functions is approximately learnable seems to justify this view.

Theorem 6 (Fulk and Jain [5]). *There is a learner M that Approx learns every recursive function.*

The following theorem answers an open question posed by Fulk and Jain [5] on whether the class of recursive functions has a learner which outputs a sequence of hypotheses that approximates the function to be learnt and almost always differs from the target only on finitely many places.

Theorem 7. *There is a learner M which learns the class of all recursive functions such that (i) M is a BC^* learner, (ii) M is a partial learner and (iii) M is an approximate learner.*

Proof. Let ψ_0, ψ_1, \dots be an enumeration of all recursive functions and some partial ones such that in every step s there is exactly one pair (e, x) for which $\psi_e(x)$ becomes defined at step s and this pair satisfies in addition that $\psi_e(y)$ is already defined by step s for all $y < x$. Furthermore, a function ψ_e is said to make progress on σ at step s iff $\psi_e(x)$ becomes defined at step s and $x \in \text{dom}(\sigma)$ and $\psi_e(y) = \sigma(y)$ for all $y \leq x$.

Now one defines for every σ a partial-recursive function $\vartheta_{e,\sigma}$ as follows:

- $\vartheta_{e,\sigma}(x) = \sigma(x)$ for all $x \in \text{dom}(\sigma)$;
- Let $e_t = e$;
- Inductively for all $s \geq t$, if some index $d < e_s$ makes progress on σ at step $s + 1$ then let $e_{s+1} = d$ else let $e_{s+1} = e_s$;
- For each value $x \notin \text{dom}(\sigma)$, if there is a step $s \geq t + x$ for which $\psi_{e_s,s}(x)$ is defined then $\vartheta_{e,\sigma}(x)$ takes this value for the least such step s , else $\vartheta_{e,\sigma}(x)$ remains undefined.

The learner M , now to be constructed, uses these functions as hypothesis space; on input τ , M outputs the index of $\vartheta_{e,\sigma}$ for the unique e and shortest prefix σ of τ such that the following three conditions are satisfied at some time t :

- t is the first time such that $t \geq |\tau|$ and some function makes progress on τ ;
- ψ_e is that function which makes progress at τ ;
- for every $d < e$, ψ_d did not make progress on τ at any $s \in \{|\sigma|, \dots, t\}$ and either $\psi_{d,|\sigma|}$ is inconsistent with σ or $\psi_{d,|\sigma|}(x)$ is undefined for at least one $x \in \text{dom}(\sigma)$.

For finitely many strings τ there might not be any such function $\vartheta_{e,\sigma}$, as τ is required to be longer than the largest value up to which some function has made progress at time $|\tau|$, which can be guaranteed only for almost all τ . For

these finitely many exceptions, M outputs a default hypothesis, e.g., for the everywhere undefined function. Now the three conditions (i), (ii) and (iii) of M are verified. For this, let ψ_d be the function to be learnt, note that ψ_d is total.

Condition (i): M is a BC^* learner. Let d be the least index of the function ψ_d to be learnt and let u be the last step where some ψ_e with $e < d$ makes progress on ψ_d . Then every $\tau \preceq \psi_d$ with $|\tau| \geq u + 1$ satisfies that first $M(\tau)$ conjectures a function $\vartheta_{e,\sigma}$ with $e \geq d$ and $|\sigma| \geq u + 1$ and $\sigma \preceq \psi_d$ and second that almost all e_s used in the definition of $\vartheta_{e,\sigma}$ are equal to d ; thus the function computed is a finite variant of ψ_d and M is a BC^* learner.

Condition (ii): M is a partial learner. Let t_0, t_1, \dots be the list of all times where ψ_d makes progress on itself with $u < t_0 < t_1 < \dots$. Note that whenever $\tau \preceq \psi_d$ and $|\tau| = t_k$ for some k then the conjecture $\vartheta_{e,\sigma}$ made by $M(\tau)$ satisfies $e = d$ and $|\sigma| = u + 1$. As none of these conjectures make progress from step $u + 1$ onwards on ψ_d , they also do not make progress on σ after step $|\sigma|$ and $\vartheta_{e,\sigma} = \psi_d$; hence the learner outputs some index for ψ_d infinitely often. Furthermore, all other indices $\vartheta_{e,\sigma}$ are output only finitely often: if $e < d$ then ψ_e makes no progress on the target function ψ_d after step u ; if $e > d$ then the length of σ depends on the prior progress of ψ_d on itself, and if $|\tau| > t_k$ then $|\sigma| > t_k$.

Condition (iii): M is an approximate learner. Conditions (d) and (e) in Proposition 5 are used. Now it is shown that, for all $\tau \preceq \psi_d$ with $t_k \leq |\tau| < t_{k+1}$, the hypothesis $\vartheta_{e,\sigma}$ issued by $M(\tau)$ is correct on the set $\{t_0, t_1, \dots\}$. If $|\tau| = t_k$ then the hypothesis is correct everywhere as shown under condition (ii). So assume that $e > d$. Then $|\tau| > t_k$ and $|\sigma| > t_k$, hence $\vartheta_{e,\sigma}(x) = \psi_d(x)$ for all $x \leq t_k$. Furthermore, as ψ_d makes progress on σ in step t_{k+1} and as no ψ_c with $c < d$ makes progress on σ beyond step $|\sigma|$, it follows that the e_s defined in the algorithm of $\vartheta_{e,\sigma}$ all satisfy $e_s = d$ for $s \geq t_{k+1}$; hence $\vartheta_{e,\sigma}(x) = \psi_d(x)$ for all $x \geq t_{k+1}$. ■

5 Approximate Learning of Languages

This section proposes three notions of approximation in language learning. The first two notions, *approximate* learning and *weak approximate* learning, are adaptations of the set of conditions for approximately learning recursive functions given in Proposition 5. Recall that a set V is a finite variant of a set W iff there is an x such that for all $y > x$ it holds that $V(y) = W(y)$.

Definition 8. Let S be a class of languages. S is *approximately (Approx) learnable* iff there is a learner M such that for every language $L \in S$ there is an infinite set W such that for all texts T and all finite variants V of W and almost all hypotheses H of M on T , $H \cap V = L \cap V$. S is *weakly approximately (WeakApprox) learnable* iff there is a learner M such that for every language $L \in S$ and for every text T for L there is an infinite set W such that for all finite variants V of W and almost all hypotheses H of M on T , $H \cap V = L \cap V$. S is *finitely approximately (FinApprox) learnable* iff there is a learner M such that for every language $L \in S$, all texts T for L , and any finite set D , it holds that for almost all hypotheses H of M on T , $H \cap D = L \cap D$.

Remark 9. Jain, Martin and Stephan [13] defined a partial-recursive function C to be an *In-classifier* for a class S of languages if, roughly speaking, for every $L \in S$, every text T for L , every finite set D , and almost all n , C on $T[n]$ will correctly “classify” all $x \in D$ as either belonging to L or not belonging to L . A learner M that *FinApprox* learns a class S may be translated into a total In-classifier for S , and vice versa.

Approximate learning requires, for each target language, the existence of a set W suitable for all texts, while in weakly approximate learning the set W may depend on T . In the weakest notion, finitely approximate learning, on any text T for a target language L the learner is only required to be almost always correct on any finite set. As will be seen later, this model is so powerful that the whole class of r.e. sets can be finitely approximated by a partial learner. The following results illustrate the models of approximate and weakly approximate learning. They establish that, in contrast to the function learning case, approximate language learnability does not imply BC^* learnability. BC^* learnability does not imply approximate learnability either, but weakly approximate learning is powerful enough to cover all BC^* learnable classes.

Proposition 10. *If there is an infinite r.e. set W such that all members of the class contain W then the class is Approx learnable.*

Proof. The learner for this just conjectures $\text{range}(\sigma) \cup W$ on any input σ . ■

Thus approximate learning does, for languages, not imply BC^* learning.⁵ Note that for infinite coinfinite r.e. sets W , the class of all r.e. supersets of W is not BC^* learnable. The next result is the mirror image of the previous result by just considering a learner which conjectures the range of the data seen so far; for each set L in the class the infinite set S in item (e) of Proposition 5 is just the complement of L .

Proposition 11. *If a class \mathcal{C} consists only of coinfinite r.e. sets then \mathcal{C} is Approx learnable.*

While the class of all coinfinite r.e. sets can be approximated, this is not true for the class of all cofinite sets.

Proposition 12. *The class of all cofinite sets is ConsWeakApprox BC^* Part learnable but neither Approx learnable nor BC^n learnable for any n .*

Proof. To make a ConsWeakApprox BC^* Part learner, define P as follows. On input σ , P determines whether or not $\text{range}(\sigma) - \text{range}(\sigma') = \{x\}$ for some $x \in \mathbb{N}$. If $\text{range}(\sigma) - \text{range}(\sigma')$ is either empty or equal to $\{\#\}$, then P repeats its last conjecture ($P(\sigma')$) if $\sigma' \neq \epsilon$; if $\sigma' = \epsilon$, then P outputs a default hypothesis, say a canonical index for \mathbb{N} . If $\text{range}(\sigma) - \text{range}(\sigma') = \{x\}$ for some $x \in \mathbb{N}$, then P determines the maximum w (if such a w exists) such that

⁵ For example, take the class of all supersets of the set of even numbers.

$w \notin \text{range}(\sigma) \cap \{0, \dots, x\}$, and outputs a canonical index for the cofinite set $(\text{range}(\sigma) \cap \{0, \dots, w\}) \cup \{z : z > w\}$. If no such w exists, then P outputs a canonical index for \mathbb{N} .

Given any text T for a cofinite set $L \neq \mathbb{N}$ such that $w = \max(\mathbb{N} - L)$, there is a sufficiently large s such that $\text{range}(T[s' + 1]) \cap \{0, \dots, w\} = L \cap \{0, \dots, w\}$ for all $s' > s$. Furthermore, there are infinitely many $n > s$ such that $\text{range}(T[n + 1]) - \text{range}(T[n]) = \{x\}$ for some number $x > w$, and on each of these text prefixes $T[n + 1]$, P will output a canonical index for L . P is also consistent by construction. Thus P consistently partially learns L . On any text T' for \mathbb{N} , there are infinitely many stages n at which $\text{range}(T'[n + 1])$ contains all numbers less than x for some x , and therefore P will output a canonical index for \mathbb{N} infinitely often. To see that P is also a *WeakApprox* learner, observe that if T'' is a text for a cofinite set L , then T'' contains an infinite subsequence $T''(n_0), T''(n_1), T''(n_2), \dots$ of numbers such that $n_0 < n_1 < n_2 < \dots$ and $T''(n_0) < T''(n_1) < T''(n_2) < \dots$, which means that for almost all n , $W_{P(T''[n])}$ contains the infinite set $\{T''(n_0), T''(n_1), T''(n_2), \dots\}$. Hence P is a *WeakApprox* learner. Note that P is also a BC^* learner as it always outputs cofinite sets.

Now assume for a contradiction that for some n and learner Q , Q BC^n learns the class of all cofinite sets. Since Q BC^n learns \mathbb{N} , there is a $\sigma \in (\mathbb{N} \cup \{\#\})^*$ such that for all $\tau \in (\mathbb{N} \cup \{\#\})^*$, $|\mathbb{N} - W_{Q(\sigma\tau)}| \leq n$. Now choose some cofinite L such that $\text{range}(\sigma) \subset L$ and $|\mathbb{N} - L| \geq 2n + 1$. Since Q must BC^n learn L , there exists some $\theta \in (L \cup \{\#\})^*$ such that $|L \Delta W_{Q(\sigma\theta)}| \leq n$. But $|\mathbb{N} - L| - |\mathbb{N} - W_{Q(\sigma\theta)}| \leq |L \Delta W_{Q(\sigma\theta)}| \leq n$, and so by the definition of σ , $|\mathbb{N} - L| \leq n + |\mathbb{N} - W_{Q(\sigma\theta)}| \leq n + n = 2n$, contradicting the definition of L . Therefore the class of all cofinite sets has no BC^n learner for any n .

Assume now that the set L to be learnt is approximated with parameter set W . Given an approximate learner M for this class, one can construct inductively a text T such that either the text is for some set $L - \{w\}$ and it conjectures almost always that w is in the set to be learnt or the text is for L while there are infinitely many conjectures which do not contain W as a subset.

The idea is to construct the text T step by step by starting in (a) below and by alternating between (a) and (b) as needed:

(a) Select a $w \in L \cap W$ not contained in the part of the text constructed so far and add to the part of the text the elements of $L - \{w\}$ in ascending order until the learner M on the so far constructed initial segment conjectures a set not containing w ;

(b) Append to the so far constructed part of the text all elements of L up to the element w (inclusively) and go back to step (a).

This gives then a text T with the desired properties: if the learner eventually stays in (a) forever, it is wrong on w considered when it the last time goes into (a); if the learner goes to (b) infinitely often, the text T is for L while the learner M conjectures infinitely often sets which are not supersets of W . Thus there is no approximate learner for the class of all cofinite sets. ■

The following result shows that weak approximate learning is quite powerful.

Theorem 13. *The class of all infinite sets is ConsWeakApprox learnable.*

Proof. Consider the learner M which conjectures on input σ the set

$$W_{M(\sigma)} = \text{range}(\sigma) \cup \{x : \forall y \in \text{range}(\sigma) [x > y]\}$$

and consider any text T for an infinite set. Let $S = \{x \in \text{range}(T) : \text{when } x \text{ appears first in } T, \text{ no larger datum of } T \text{ has been seen so far}\}$. Note that the set S is infinite. Now all conjectures $M(T[n])$ are a superset of S : if an $x \in S$ has not yet appeared in $T[n]$ then all members of $\text{range}(T[n])$ are smaller than x and $x \in W_{M(T[n])}$ else x has already appeared in $T[n]$ and is therefore also in $\text{range}(T[n])$. Furthermore, if $x \notin \text{range}(T)$ then almost all n satisfy $\max(\text{range}(T[n])) > x$ and therefore $x \notin W_{M(T[n])}$, thus for every x almost all hypotheses $W_{M(T[n])}$ are correct at x . ■

Unfortunately, the weakly approximate learning property of any class of infinite sets may be lost if finite sets are added to the target class.

Proposition 14. *Gold's class consisting of the set of natural numbers and all sets $\{0, 1, \dots, m\}$ is not WeakApprox learnable.*

Proof. Make a text T where $T(0) = 0$ and iff the n -th hypothesis of the learner contains $T(n) + 1$ then $T(n + 1) = T(n)$ else $T(n + 1) = T(n) + 1$.

In the case that the text T is for a finite set with maximum m then $T(n) = m$ for almost all n and the n -th hypothesis contains $m + 1$ for almost all n ; thus the approximations are in the limit false at $m + 1$.

In the case that the text T is for the set of all natural numbers then consider any $m > 0$ and consider the first n such that $T(n + 1) = m$. Then the n -th hypothesis does not contain m . Therefore, one can conclude that for every m there is an $n \geq m$ such that the n -th hypothesis is conjecturing m not to be in the set to be learnt although the set to be learnt is the set of all natural numbers. In particular there is no infinite set on which from some time on all approximations are correct.

Thus the class considered is not weakly approximately learnable. ■

It may be observed that in the proof of Theorem 13, the parameter sets S with respect to which the learner M approximates the class of all infinite sets may not necessarily be r.e. (or be of any fixed Turing degree). This motivates the question of whether or not the class of all infinite sets is still weakly approximately learnable if one restricts the class of parameter sets in Definition 8 to some countable family.

Definition 15. For any sets L and W , where W is infinite, and any text T for L , say that a recursive learner M *weakly approximately (WeakApprox) learns L via W on T* iff for all finite variants V of W , it holds that for almost all hypotheses H of M on T , $H \cap V = L \cap V$. For any class \mathcal{W} of infinite sets, a class S of sets is *weakly approximately (WeakApprox) learnable via \mathcal{W}* iff there is a recursive learner M such that for every $L \in S$ and every text T for L , M *WeakApprox* learns L via some $W \in \mathcal{W}$ on T .

Proposition 16. *For any countable class \mathcal{W} of infinite sets, the class of all cofinite sets is not WeakApprox learnable via \mathcal{W} .*

Proof. Suppose M is a recursive learner that weakly approximately learns all cofinite sets via some countable class \mathcal{W} of infinite sets. First, note that there exist $\sigma \in \mathbb{N}^*$ and $V \in \mathcal{W}$ such that for all $\tau \in \mathbb{N}^*$, $V \subseteq W_{M(\sigma\tau)}$. For, assuming otherwise, one can build a text T for \mathbb{N} as follows. Let V_0, V_1, V_2, \dots be a one-one enumeration of \mathcal{W} , and set $T_0 = \epsilon$, where T_s denotes the text prefix built until stage s . Let m_s be the minimum number not contained in $\text{range}(T_s)$, and find strings $\eta_0, \eta_1, \dots, \eta_s$ such that for all $i \in \{0, \dots, s\}$, $V_i \not\subseteq W_{M(T_s \circ m_s \eta_0 \dots \eta_i)}$; by assumption, such strings $\eta_0, \eta_1, \dots, \eta_s$ must exist. Let $T = \lim_s T_s$. T is a text for \mathbb{N} ; furthermore, for any $V_i \in \mathcal{W}$, $V_i \not\subseteq W_{M(T[s+1])}$ for infinitely many s , so that M does not weakly approximately learn \mathbb{N} via V_i on T .

Now fix $\sigma \in \mathbb{N}^*$ and $V \in \mathcal{W}$ such that for all $\tau \in \mathbb{N}^*$, $V \subseteq W_{M(\sigma\tau)}$. As V is infinite, one can choose some $w \in V - \text{range}(\sigma)$. Let T' be a text for $\mathbb{N} - \{w\}$ that extends σ . Then M conjectures a set containing w on almost all text prefixes of T' , which shows that it cannot weakly approximately learn $\mathbb{N} - \{w\}$. In conclusion, the class of all cofinite sets is not weakly approximately learnable via \mathcal{W} . ■

Theorem 17. *If \mathcal{C} is BC^* learnable then \mathcal{C} is WeakApprox learnable.*

Proof. By Theorem 13, there is a learner M that weakly approximates the class of all infinite sets. Let O be a BC^* learner for \mathcal{C} . Now the new learner N is given as follows: On input σ , $N(\sigma)$ outputs an index of the following set which first enumerates $\text{range}(\sigma)$ and then searches for some τ that satisfies the following conditions: (1) $\text{range}(\tau) = \text{range}(\sigma)$; (2) $|\tau| = 2 * |\text{range}(\sigma)|$; (3) $W_{O(\tau\#^s)}$ enumerates at least $|\sigma|$ many elements for all $s \leq |\sigma|$. If all three conditions are met then the set contains also all elements of $W_{M(\sigma)}$. If $L \in \mathcal{C}$ is finite then for every τ of length $2 * |L|$ with range L , the learner outputs on some input $\tau\#^{s_\tau}$ a finite set with c_τ many elements. As there are only finitely many such τ , there is an upper bound t of all c_τ and s_τ . Then it follows from the construction that the learner N on any input σ with $\text{range}(\sigma) = L$ and $|\sigma| \geq t$ outputs a hypothesis for the set L , as the corresponding τ cannot be found. Thus N weakly approximately learns L .

If $L \in \mathcal{C}$ is infinite then there is a locking sequence $\gamma \in L^*$ for L such that $O(\gamma\eta)$ conjectures an infinite set whenever $\eta \in L^*$. It follows for all σ with $\text{range}(\gamma) \subseteq \text{range}(\sigma)$ and $|\text{range}(\sigma)| > |\gamma|$ that $N(\sigma)$ considers also a τ which is an extension of γ in its algorithm and which therefore meets all three conditions, thus $N(\sigma)$ will conjecture a set consisting of the union of $\text{range}(\sigma)$ and $W_{M(\sigma)}$. As adding $\text{range}(\sigma)$ to the hypothesis $W_{M(\sigma)}$ cannot make $W_{N(\sigma)}$ to be incorrect at any x where $W_{M(\sigma)}$ is correct, it follows that also N is weak approximately learning L . Thus, by case distinction, N is a weak approximate learner for \mathcal{C} . ■

6 Combining Partial Language Learning With Variants of Approximate Learning

This section is concerned with the question whether partial learners can always be modified to approximate the target language in the models introduced above.

6.1 Finitely Approximate Learning

The first results demonstrate the power of the model of finitely approximate learning: there is a partial learner that finitely approximates every r.e. language.

Theorem 18. *The class of all r.e. sets is FinApproxPart learnable.*

Proof. Let M_1 be a partial learner of all r.e. sets. Define a learner M_2 as follows. Given a text T , let $e_n = M_1(T[n+1])$ for all n . On input $T[n+1]$, M_2 determines the finite set $D = \text{range}(T[n+1]) \cap \{0, \dots, m\}$, where m is the minimum $m \leq n$ with $e_m = e_n$. M_2 then outputs a canonical index for $D \cup (W_{e_n} \cap \{x : x > m\})$.

Suppose T is a text for some r.e. set L . Then there is a least l such that M_1 on T outputs e_l infinitely often and $W_{e_l} = L$. Furthermore, there is a least l' such that for all $l'' > l'$, $D_{l''} = \text{range}(T[l''+1]) \cap \{0, \dots, l\} = L \cap \{0, \dots, l\}$. Hence M_2 will output a canonical index for $L = D_{l''} \cup (W_{e_{l''}} \cap \{x : x > l\})$ infinitely often. On the other hand, since, for every h with $e_h \neq e_l$ and $e_h \neq e_{h'}$ for all $h' < h$, M_1 outputs e_h only finitely often, M_2 will conjecture sets of the form $D' \cup (W_{e_h} \cap \{x : x > h\})$ only finitely often. Thus M_2 partially learns L .

To see that M_2 is also a finitely approximate learner, consider any number x . Suppose that M_1 on T outputs exactly one index e infinitely often; further, $W_e = L$ and j is the least index such that $e_j = e$. Let s be sufficiently large so that for all $s' > s$, $\text{range}(T[s'+1]) \cap \{0, \dots, \max(\{x, j\})\} = L \cap \{0, \dots, \max(\{x, j\})\}$. First, assume that M_1 outputs only finitely many distinct indices on T . It follows that M_1 on T converges to e . Thus M_2 almost always outputs a canonical index for $(L \cap \{0, \dots, j\}) \cup (W_{e_j} \cap \{y : y > j\})$, and so it approximately learns L . Second, assume that M_1 outputs infinitely many distinct indices on T . Let d_1, \dots, d_x be the first x conjectures of M_1 that are pairwise distinct and are not equal to e . There is a stage $t > s$ large enough so that $e_{t'} \notin \{d_1, \dots, d_x\}$ for all $t' > t$. Consequently, whenever $t' > t$, M_2 on $T[t'+1]$ will conjecture a set W such that $W \cap \{0, \dots, x\} = L \cap \{0, \dots, x\}$. This establishes that M_2 finitely approximately learns any r.e. set. ■

It may be observed in the proof of Theorem 18 that if M_1 is a confident partial learner of some class \mathcal{C} , then M_2 confidently partially as well as finitely approximately learns \mathcal{C} . This observation leads to the next theorem.

Theorem 19. *If \mathcal{C} is ConfPart learnable, then \mathcal{C} is FinApproxConfPart learnable.*

Gao, Jain and Stephan [7] showed that consistently partial learners exist for all and only the subclasses of uniformly recursive families; the next theorem shows that such learners can even be finitely approximate at the same time, in addition to being *prudent*.

Theorem 20. *If \mathcal{C} is a uniformly recursive family, then \mathcal{C} is *FinApproxConsPart* learnable by a prudent learner.*

Proof. Let $\mathcal{C} = \{L_0, L_1, L_2, \dots\}$ be a uniformly recursive family. On text T , define M at each stage s as follows:

- If there are $x \in \mathbb{N}$ and $i \in \{0, 1, \dots, s\}$ such that
- $\text{range}(T[s+1]) - \text{range}(T[s]) = \{x\}$,
 - $\text{range}(T[s+1]) \subseteq L_i \cup \{\#\}$ and
 - $\text{range}(T[s+1]) \cap \{0, \dots, x\} = L_i \cap \{0, \dots, x\}$

Then M outputs the least such i

Else M outputs a canonical index for $\text{range}(T[s+1]) - \{\#\}$.

The consistency of M follows directly by construction. If T is a text for a finite set then the “Else-Case” will apply almost always and M converges to a canonical index for $\text{range}(T)$. Now consider that T is a text for some infinite set $L_m \in \mathcal{C}$ and m is the least index of itself. Let t be large enough so that for all $t' > t$, all $x \in L - \text{range}(T[t+1]) - \{\#\}$ and all $j < m$, $L_j \cap \{0, \dots, x\} \neq \text{range}(T[t'+1]) \cap \{0, \dots, x\}$. There are infinitely many stages $s > \max(\{t, m\})$ at which $T(s) \notin \text{range}(T[s]) \cup \{\#\}$ and $\text{range}(T[s+1]) \cap \{0, \dots, T(s)\} = L \cap \{0, \dots, T(s)\}$. At each of these stages, M will conjecture L_m . Thus M conjectures L_m infinitely often. Furthermore, for every x there is some s_x such that for all $y \in L - \text{range}(T[s_x+1])$, it holds that $y > x$. Thus whenever $s' > s_x$, M ’s conjecture on $T[s'+1]$ agrees with L on $\{0, \dots, x\}$. M is therefore a finitely approximate learner, implying that it never conjectures any incorrect index infinitely often. ■

Proposition 20 and [8, Theorem 18] together give the following corollary.

Corollary 21. *If \mathcal{C} is *ConsPart* learnable, then \mathcal{C} is *FinApproxConsPart* learnable by a prudent learner.*

The following result shows that also conservative partial learning may always be combined with finitely approximate learning.

Theorem 22. *If \mathcal{C} is *ConsvPart* learnable, then \mathcal{C} is *FinApproxConsvPart* learnable.*

Proof. Let M_1 be a *ConsvPart* learner for \mathcal{C} , and suppose that M_1 outputs the sequence of conjectures e_0, e_1, \dots on some given text T . The construction of a new learner M_2 is similar to that in Theorem 18; however, one has to ensure that M_2 does not output more than one index that is either equal to or a proper superset of the target language. On input $T[s+1]$, define $M_2(T[s+1])$ as follows.

1. If $\text{range}(T[s+1]) \subseteq \{\#\}$ then output a canonical index for \emptyset else go to 2.
2. Let $m \leq s$ be the least number such that $e_m = e_s$. If $W_{e_s, s} \cap \{0, \dots, m\} = \text{range}(T[s+1]) \cap \{0, \dots, m\} = D$ then output a canonical index for $D \cup (W_{e_m} \cap \{x : x > m\})$ else go to 3.
3. If $s \geq 1$ then output $M_2(T[s])$ else output a canonical index for \emptyset .

Suppose that T is a text for some $L \in \mathcal{C}$. Without loss of generality, assume that $L \neq \emptyset$; if $L = \emptyset$, then M_2 will always output a canonical index for \emptyset . M_1 on T outputs exactly one index e_h infinitely often, where $W_{e_h} = L$ and $e_{h'} \neq e_h$ for all $h' < h$. Let s be the least stage at which $\text{range}(T[s+1]) \cap \{0, \dots, h\} = L \cap \{0, \dots, h\} = W_{e_h, s} \cap \{0, \dots, h\}$. Then for all $s' \geq s$ such that $e_{s'} = e_h$, step 2. will apply, so that M_2 outputs a canonical index g for $(L \cap \{0, \dots, h\}) \cup (W_{e_h} \cap \{x : x > h\}) = L$. Since there are infinitely many such s' , M_2 will output g infinitely often. Consider any other set of the form $F \cup (W_{e_l} \cap \{x : x > l\})$ that M_2 may conjecture at some stage t , where $l \neq h$ and $e_{l'} \neq e_l$ for all $l' < l$. By construction, F is equal to $W_{e_l, t} \cap \{0, \dots, l\}$. Thus $F \cup (W_{e_l} \cap \{x : x > l\}) \subseteq W_{e_l}$, and so by the partial conservativeness of M_1 , $L \not\subseteq F \cup (W_{e_l} \cap \{x : x > l\})$. If M_2 conjectures some set of the form $G \cup (W_{e_h} \cap \{x : x > h\})$, where $G \neq L \cap \{0, \dots, h\}$, then there is some $y \in L - (G \cup (W_{e_h} \cap \{x : x > h\}))$, and so $L \not\subseteq G \cup (W_{e_h} \cap \{x : x > h\})$. Furthermore, $L \not\subseteq \emptyset$. Therefore M_2 outputs exactly one index for a set that contains L , and M_2 outputs this index infinitely often. To show that M_2 outputs any incorrect index only finitely often, it is enough to show that it finitely approximately learns L .

Consider any x . If M_1 on T outputs only finitely many distinct indices, then one can argue as in Theorem 18 that M_2 converges on T to g . Suppose that M_1 on T outputs infinitely many distinct indices. Let s be the least stage at which $\text{range}(T[s+1]) \cap \{0, \dots, x\} = L \cap \{0, \dots, x\}$. Let d_1, \dots, d_x be x pairwise distinct indices of M_1 on T , none of which is equal to e_h . Then there is a least stage $t > s$ such that $M_2(T[t+1]) = g$ and for all $t' > t$, $e_{t'} \notin \{d_1, \dots, d_x\}$. Thus on any $T[t'+1]$ with $t' > t$, M_2 either outputs g or conjectures a set W such that $W \cap \{0, \dots, x\} = L \cap \{0, \dots, x\}$. Therefore M_2 is both a finitely approximate and a conservatively partial learner of \mathcal{C} . ■

Jain, Stephan and Ye [12] proved that for uniformly r.e. classes, class-comprising explanatory learning is equivalent to *uniform* explanatory learning; the latter means that one can construct a numbering of partial-recursive learners M_0, M_1, M_2, \dots such that for any given r.e. numbering H_0, H_1, H_2, \dots of the target class \mathcal{C} with $W_e = \{\langle d, x \rangle : x \in H_d\}$, the e -th learner explanatorily learns \mathcal{C} with respect to $\{H_0, H_1, H_2, \dots\}$. In particular, uniformly r.e. explanatorily learnable classes are always explanatorily learnable with respect to a class-preserving hypothesis space. The next theorem shows, however, that none of the approximate learning criteria considered so far can be combined with class-preservingness. Thus, in general, any successful approximation of languages must involve sets not contained in the target hypothesis space. An intuitive explanation for this is that a class-preserving learner may be incapable of recursively deciding, for any given finite set D , whether there exists a language in the target class that agrees with the current input on D .

Theorem 23. *There is a uniformly r.e. class that is Approx learnable but not ClsPresvFinApprox learnable.*

Proof. Let M_0, M_1, M_2, \dots be an enumeration of all partial-recursive learners. For each e , define a strictly increasing r.e. sequence $x_{e,1}, x_{e,2}, \dots$ as follows.

First, for any given finite set D and number $y \notin D$, let $\alpha_{D,y}$ denote the string $1 \circ 4 \circ \dots \circ 3y + 1$, which is a concatenation (in increasing order) of all numbers of the form $3z + 1$ with $0 \leq z \leq y$ and $z \notin D$. $x_{e,1}$ is defined to be the first number found (if such a number exists) such that for some $m_{e,1}$ with $x_{e,1} > m_{e,1}$, it holds that $\{3e, 3x_{e,1} + 1\} \subseteq W_{M_e(3e \circ \alpha_{\emptyset, m_{e,1}})}$. Suppose that $x_{e,1}, \dots, x_{e,k}$ have been defined. $x_{e,k+1}$ is then defined to be the first number found (if such a number exists) such that for some $m_{e,k+1}$ with $x_{e,k+1} > m_{e,k+1} > x_{e,k}$, it holds that $\{3e, 3x_{e,k+1} + 1\} \subseteq W_{M_e(3e \circ \alpha_{\{x_{e,i} : 1 \leq i \leq k\}, m_{e,k+1}}})}$.

For each pair $\langle e, i \rangle$, define $L_{\langle e, i \rangle}$ according to the following case distinction.

Case (1): $x_{e,i}$ is defined for all i . Set $L_{\langle e, 0 \rangle} = \{e\} \oplus (\mathbb{N} - \{x_{e,i} : i \in \mathbb{N}\}) \oplus \emptyset$. For each $j > 0$, set $L_{\langle e, j \rangle} = \{e\} \oplus (\mathbb{N} - \{x_{e,i} : i < j\}) \oplus \{0\}$.

Case (2): There is a minimum l such that $x_{e,l}$ is undefined. Set $L_{\langle e, 0 \rangle} = \{e\} \oplus (\{y : (l = 1 \Rightarrow y < 0) \wedge (l > 1 \Rightarrow y < x_{e,l-1})\} - \{x_{e,i} : i < l\}) \oplus \emptyset$. For each j with $1 \leq j \leq l - 1$, set $L_{\langle e, j \rangle} = \{e\} \oplus (\mathbb{N} - \{x_{e,i} : i < j\}) \oplus \{0\}$. Set $L_{\langle e, l \rangle} = \{e\} \oplus (\mathbb{N} - \{x_{e,i} : i < l\}) \oplus \emptyset$. For each $j \geq l + 1$, set $L_{\langle e, j \rangle} = \emptyset$.

Set $\mathcal{C} = \{L_{\langle e, i \rangle} : e, i \in \mathbb{N}\}$.

Now it is shown that \mathcal{C} is approximately learnable with respect to a class-comprising hypothesis space. On input σ , the learner M outputs a canonical index for \emptyset if $\text{range}(\sigma)$ does not contain any multiple of 3. Otherwise, let e be the minimum number such that $3e \in \text{range}(\sigma)$; M then checks whether or not $2 \in \text{range}(\sigma)$. If $2 \in \text{range}(\sigma)$, M searches (with computational time bounded by $|\sigma|$) for the least l (if such an l exists) such that $3x_{e,l} + 1 \in \text{range}(\sigma)$; it then conjectures $L_{\langle e, l \rangle}$. If no such l exists, M conjectures $L_{\langle e, 1 \rangle}$. If $2 \notin \text{range}(\sigma)$, M searches for the minimum l' such that $x_{e,l'}$ has not yet been defined at stage $|\sigma|$. If $3x_{e,l'} + 1 \notin \text{range}(\sigma)$, then M conjectures $L_{\langle e, 0 \rangle}$. If $3x_{e,l'} + 1 \in \text{range}(\sigma)$, then M outputs an index d such that

$$W_d = \begin{cases} \text{range}(\sigma) \cup \{3z + 1 : (l' = 1 \Rightarrow 0 \leq z \leq s) \text{ if } s > x_{e,l'} \text{ is the first step at} \\ \wedge (l' > 1 \Rightarrow x_{e,l'-1} + 1 \leq z \leq s)\} \cup L_{\langle e, 0 \rangle} & \text{which } x_{e,l'} \text{ is defined;} \\ \text{range}(\sigma) \cup \{3z + 1 : (l' = 1 \Rightarrow z \geq 0) & \text{if } x_{e,l'} \text{ is undefined.} \\ \wedge (l' > 1 \Rightarrow z \geq x_{e,l'-1} + 1)\} \end{cases}$$

For the verification that M approximately learns \mathcal{C} , suppose that M outputs the sequence of conjectures e_0, e_1, e_2, \dots on text T . Assume first that $x_{e,i}$ is defined for all i . If T is a text for $L_{\langle e, 0 \rangle}$, then for almost all n , W_{e_n} is a finite variant of $L_{\langle e, 0 \rangle}$; furthermore, if e_{j_0}, e_{j_1}, \dots is the subsequence of conjectures for which $W_{e_{j_i}} \neq L_{\langle e, 0 \rangle}$, then the sequence y_0, y_1, y_2, \dots of minimum numbers such that $W_{e_{j_i}}(y_i) \neq L_{\langle e, 0 \rangle}(y_i)$ is almost always monotone increasing and contains a strictly increasing subsequence. In addition, for almost all i , $W_{e_i}(y) = L_{\langle e, 0 \rangle}(y)$ for all y contained in $L_{\langle e, 0 \rangle}$, which is an infinite set. Hence M approximately learns $L_{\langle e, 0 \rangle}$. If T is a text for $L_{\langle e, j \rangle}$ for some $j > 0$, then $2 \in \text{range}(T)$ and so M will eventually identify j as the minimum l such that $3x_{e,l} + 1 \in \text{range}(T)$. Thus M will converge to an index for $L_{\langle e, j \rangle}$. Next, assume that there is a minimum l such that $x_{e,l}$ is undefined. If T is a text for $L_{\langle e, 0 \rangle}$, then M will in the limit identify l as the minimum l' such that $x_{e,l'}$ is undefined; thus, as $3x_{e,l'} + 1 \notin$

$\text{range}(T)$, M on T will converge to an index for $L_{\langle e,0 \rangle}$. If T is a text for some nonempty $L_{\langle e,j \rangle}$ with $j > 1$, M on T will again converge to an index for $L_{\langle e,j \rangle}$: if $2 \in L_{\langle e,j \rangle}$, then M will eventually identify j as the minimum number l such that $3x_{e,l} + 1 \in \text{range}(T)$ and converge to indices for $L_{\langle e,j \rangle}$; if $2 \notin L_{\langle e,j \rangle}$, then $3x_{e,j} + 1 \in \text{range}(T)$ and the fact that j is the minimum number for which $x_{e,j}$ is undefined together imply that M on T will converge to indices for $L_{\langle e,j \rangle}$. By construction, M converges to a canonical index for \emptyset on any text with an empty range. This completes the verification that M approximately learns \mathcal{C} .

It remains to show that \mathcal{C} is not *FinApprox* learnable using a class-preserving hypothesis space. Assume that M_e *ClspresvFinApprox* learns \mathcal{C} . If there is a minimum l such that $x_{e,l}$ is undefined, then there is a text U for $L_{\langle e,l \rangle}$ on which M_e almost always outputs a conjecture that is different from $L_{\langle e,l \rangle}$. Since M_e finitely approximates $L_{\langle e,l \rangle}$, almost all of M_e 's hypotheses on U must contain $3e$. But for all $j > 0$ such that $j \neq l$, either $L_{\langle e,j \rangle} = \emptyset$ or $2 \in L_{\langle e,j \rangle}$. As $2 \notin L_{\langle e,l \rangle}$ and $L_{\langle e,l \rangle}$ is infinite, while $L_{\langle e,0 \rangle}$ is finite, it follows that M_e , being a finitely approximate learner, must almost always conjecture a set different from any $L_{\langle e,j \rangle}$ with $j \neq l$. Hence M_e is not a finitely approximate learner of $L_{\langle e,l \rangle}$. Suppose, on the other hand, that $x_{e,i}$ is defined for all i . Then one can build a text U' for $L_{\langle e,0 \rangle}$ on which M_e infinitely often conjectures a set containing 2; but since $2 \notin L_{\langle e,0 \rangle}$, it follows that M_e does not finitely approximately learn $L_{\langle e,0 \rangle}$. This establishes that \mathcal{C} is not *ClspresvFinApprox* learnable. ■

The main content of the following proposition may be summed up as follows: the quality of the hypotheses issued by a BC^* learner may be improved so that for any given finite set D , the learner's hypotheses will eventually agree with the target language on D .

Proposition 24. *If \mathcal{C} is BC^* learnable, then \mathcal{C} is *FinApprox* BC^* learnable.*

Proof. Given a BC^* learner M of \mathcal{C} , one can make a new learner N as follows. On input σ , N conjectures $\text{range}(\sigma) \cup (W_{M(\sigma)} \cap \{z : z > |\sigma|\})$. Suppose that N is fed with a text T for some $L \in \mathcal{C}$. N is a BC^* learner because it always conjectures finite variants of M 's conjectures. Furthermore, for every finite set D there is some s_D such that $s_D > \max(D)$ and $\text{range}(T[s]) \cap D = L \cap D$ for all $s > s_D$. It follows by construction that for all $s > s_D$, $W_{N(T[s])} \cap D = \text{range}(T[s]) \cap D = L \cap D$, and so N finitely approximately BC^* learns L . ■

The next two results consider combinations of finite approximation and some learning models that permit finitely many anomalies. It is readily seen that the additional constraint of finite approximation implies that any anomaly in the learner's hypotheses will eventually be corrected.

Proposition 25. *If \mathcal{C} is Vac^* *FinApprox* learnable, then \mathcal{C} is *Vac* learnable.*

Proposition 26. *If \mathcal{C} is Ex^* *FinApprox* learnable, then \mathcal{C} is *Ex* learnable.*

As a side remark, *ConsvPartBC* learning is only as powerful as *ConsvEx* learning; the following proposition establishes this fact.

Proposition 27. *If \mathcal{C} is $ConsvPartBC$ learnable, then \mathcal{C} is $PrudConsvEx$ learnable.*

Proof. Note that on any text for some $L \in \mathcal{C}$, a $ConsvPartBC$ learner M outputs exactly one index e with $W_e = L$; since M is also a BC learner, this means that M on T converges to e and it never outputs a proper superset of L . By [8, Theorem 29] and [7, Theorem 10], \mathcal{C} is $PrudConsvEx$ learnable. ■

6.2 Weakly Approximate, Approximate and BC^* Learning

The next proposition shows that Theorem 20 cannot be improved and gives a negative answer to the question whether partial or consistent partial learning can be combined with weakly approximate learning.

Proposition 28. *The uniformly recursive class $\{A : A = \mathbb{N} \text{ or } A \text{ contains all even and finitely many odd numbers or } A \text{ contains finitely many even and all odd numbers}\}$ is (a) $ConsWeakApprox$ learnable and (b) $ConsPart$ learnable, but not $WeakApproxPart$ learnable.*

Proof. That (a) can be satisfied follows from Theorem 13; that (b) can be satisfied follows from [8, Theorem 18]. Furthermore, one can easily make a text T which makes sure that a given partial learner M for the class does not also weakly approximate it. The idea is to define the text T inductively as follows by going through the following loop:

1. Let $n = 0$;
2. As long as $M(T[n])$ does not conjecture a set which contains all even numbers and only finitely many odd numbers let $T(n)$ be the least even number not yet in the text and update $n = n + 1$;
3. As long as $M(T[n])$ does not conjecture a set which contains all odd numbers and only finitely many even numbers let $T(n)$ be the least odd number not yet in the text and update $n = n + 1$;
4. Go to Step 2.

It is easy to see that as the learner is partial it cannot get stuck in Step 2 or Step 3 forever, as it would not output an index for $range(T)$ infinitely often in that case. Hence it alternates between Steps 2 and 3 infinitely often and will therefore alternate between sets containing all even and only finitely many odd numbers and all odd and only finitely many even numbers. Hence there is no infinite set which is contained in almost all hypotheses; however, the range of T is the set of natural numbers and thus the learner is not weakly approximating it. ■

The next theorem shows that neither partial learning nor consistent partial learning can be combined with approximate learning. In fact, it establishes a stronger result: consistent partial learnability and approximate learnability are insufficient to guarantee both partial and weakly approximate learnability simultaneously.

Theorem 29. *There is a class of r.e. sets with the following properties:*

- (i) *The class is not BC^* learnable;*
- (ii) *The class is not $WeakApproxPart$ learnable;*
- (iii) *The class is $Approx$ learnable;*
- (iv) *The class is $Ex[K']$ learnable.*
- (iv) *The class is $ConsPart$ learnable.*

Proof. The key idea is to diagonalise against a list M_0, M_1, \dots of learners which are all total and which contains for every learner to be considered a delayed version. This permits to ignore the case that some learner is undefined on some input.

The class witnessing the claim consists of all sets L_d such that for each d , either L_d is $\{d, d+1, \dots\}$ or L_d is a subset built by the following diagonalisation procedure: One assigns to each number $x \geq d$ a level $\ell(x)$.

- If some set $L_{d,e} = \{x \geq d : \ell(x) \leq e\}$ is infinite then let $L_d = L_{d,e}$ for the least such e and M_d does not partially learn L_d
- else let $L_d = \{d, d+1, \dots\}$ and M_d does not weakly approximate L_d .

The construction of the sets is inductive over stages. For each stage $s = 0, 1, 2, \dots$:

- Let τ_e be a sequence of all $x \in \{d, d+1, \dots, d+s-1\}$ with $\ell(x) = e$ in ascending order;
- If there is an $e < s$ such that e has not been cancelled in any previous step and for each $\eta \preceq \tau_e$ the intersection $W_{M_d(\tau_0\tau_1\dots\tau_{e-1}\eta),s} \cap \{y : d \leq y < d+s \wedge \ell(y) > e\}$ contains at least $|\tau_e|$ elements
 - Then choose the least such e and let $\ell(d+s) = e$ and cancel all e' with $e < e' \leq s$
 - Else let $\ell(d+s) = s$.

A text $T = \lim_e \sigma_e$ is defined as follows (where σ_0 is the empty sequence):

- Let τ_e be the sequence of all x with $\ell(x) = e$ in ascending order;
- If σ_e is finite then let $\sigma_{e+1} = \sigma_e \tau_e$ else let $\sigma_{e+1} = \sigma_e$.

In case some σ_e are infinite, let e be smallest such that σ_e is infinite. Then $T = \sigma_e$ and $L_d = L_{d,e}$ and T is a text for L_d . As $L_{d,e}$ is infinite, one can conclude that

$$\forall \eta \preceq \sigma_e \forall c [|W_{M_d(\tau_0\tau_1\dots\tau_{e-1}\eta)} \cap \{y : \ell(y) > e\}| \geq c]$$

and thus M_d outputs on T almost always a set containing infinitely many elements outside L_d ; so M_d does neither partially learn L_d nor BC^* learn L_d .

In case all σ_e are finite and therefore all $L_{d,e}$ are finite there must be infinitely many e that never get cancelled. Each such e satisfies

$$\exists \eta \preceq \tau_e [W_{M_d(\tau_0\tau_1\dots\tau_{e-1}\eta)} \cap \{y : \ell(y) > e\} \text{ is finite}]$$

and therefore e also satisfies $\exists \eta \preceq \tau_e [W_{M_d(\tau_0\tau_1\dots\tau_{e-1}\eta)} \text{ is finite}]$. Thus M_d outputs on the text T for the cofinite set $L_d = \{d, d+1, \dots\}$ infinitely often a finite set

and M_d is neither weakly approximately learning L_d (as there is no infinite set on which almost all conjectures are correct) nor BC^* -learning L_d . Thus claims (i) and (ii) are true.

Next it is shown that the class of all L_d is approximately learnable by some learner N . This learner N will on a text for L_d eventually find the minimum d needed to compute the function ℓ . Once N has found this d , N will on each input σ conjecture the set

$$W_{N(\sigma)} = \{x : x \geq \max(\text{range}(\sigma)) \vee \exists y \in \text{range}(\sigma) [\ell(x) \leq \ell(y)]\}$$

In case $L_d = L_{d,e}$ for some e , $L_{d,e}$ is infinite, and for each text for $L_{d,e}$, almost all prefixes σ of this text satisfy $\max\{\ell(y) : y \in \text{range}(\sigma)\} = e$ and $L_{d,e} \subseteq W_{N(\sigma)}$. So almost all conjectures are correct on the infinite set L_d itself. Furthermore, $W_{N(\sigma)}$ does not contain any $x < \max(\text{range}(\sigma))$ with $\ell(x) > e$, hence N eventually becomes correct also on any $x \notin L_{d,e}$ and therefore N approximates $L_{d,e} = L_d$.

In case $L_d = \{d, d+1, \dots\}$, all $L_{d,e}$ are finite. Then consider the infinite set $S = \{x : \forall y > x [\ell(y) > \ell(x)]\}$. Let $x \in S$ and consider any σ with $\min(\text{range}(\sigma)) = d$. If $x \geq \max(\text{range}(\sigma))$ then $x \in W_{N(\sigma)}$. If $x < \max(\text{range}(\sigma))$ then $\ell(\max(\text{range}(\sigma))) \geq \ell(x)$ and again $x \in W_{N(\sigma)}$. Thus $W_{N(\sigma)}$ contains S . Furthermore, for all $x \geq d$ and sufficiently long prefixes σ of the text, $\ell(\max(\text{range}(\sigma))) \geq \ell(x)$ and therefore all $x \in W_{N(\sigma)}$ for almost all prefixes σ of the text. So again N approximates L_d . Thus claim (iii) is true.

Furthermore, there is a K' -recursive learner O which explanatorily learns the class. On input σ with at least one element in $\text{range}(\sigma)$, the learner determines $d = \min(\text{range}(\sigma))$. If there is now some $e \leq |\sigma|$ such that $L_{d,e}$ is infinite then O conjectures $L_{d,e}$ for the least such e else O conjectures $\{d, d+1, \dots\}$. It is easy to see that these hypotheses converge to the set L_d to be learnt: eventually the minimum of the range of each input is d . In the case that $L_d = L_{d,e}$ for some e this e is detected whenever the input is longer than e and therefore the learner converges to $L_{d,e}$. In the case that all $L_{d,e}$ are finite, the learner almost always outputs the same hypothesis for $\{d, d+1, \dots\}$. Thus O is a $Ex[K']$ learner and condition (iv) is true.

It remains to show that the class is *ConsPart* learnable. This follows from the fact that the class is a subclass of the uniformly recursive family $\mathcal{U} = \{L_{e,d}\}_{e,d \in \mathbb{N}} \cup \{\{d+x : x \in \mathbb{N}\} : d \in \mathbb{N}\}$. To see that \mathcal{U} is uniformly recursive, it may be observed from the construction of $L_{e,d}$ that for each d , $\ell(x)$ is defined for all $x \geq d$; each of these values, moreover, can be calculated effectively. Thus one can uniformly decide for all d, e and y whether or not $y \geq d$ and $\ell(y) \leq e$, that is, whether or not $y \in L_{e,d}$. Consequently, by [8, Theorem 18], the given class is consistently partially learnable, as required. ■

The next result separates *ConsApproxPart* learning from BC^* learning.

Proposition 30. *The class $\mathcal{C} = \{\mathbb{N}\} \cup \{\{0, \dots, e\} \cup \{2x : 2x > e\} : e \in \mathbb{N}\}$ is *ConsApproxPart* learnable but not BC^* learnable.*

Proof. Make a learner M as follows. On input σ , if $\text{range}(\sigma) - \text{range}(\sigma') = \{x\}$ for some odd number x , then M outputs a canonical index for \mathbb{N} . Otherwise,

M determines the maximum odd number d (if such a d exists) such that $d \in \text{range}(\sigma)$, and outputs a canonical index for $\{y : y \leq d\} \cup \{2z : 2z > d\}$. If no such d exists, then M outputs a canonical index for the set of all even numbers. Note that M is consistent by construction. If M is fed with a text T for some set $L = \{0, \dots, e\} \cup \{2x : 2x > e\}$, then there is a least s such that $\{0, \dots, e\} \subseteq \text{range}(T[s])$. Thus for all $s' \geq s$, M will output a canonical index for L and so it explanatorily learns L . If M is fed with a text for \mathbb{N} , then it will output a canonical index for \mathbb{N} at all stages where a new odd number appears; that is, it will output a canonical index for \mathbb{N} infinitely often. Furthermore, since M 's conjecture at every stage contains the set of all even numbers, and $\{0, \dots, f\}$ is contained in almost all of M 's conjectures for every f , M is an approximate learner, which implies that it never outputs any incorrect index infinitely often. Hence M *ConsApproxPart* learns \mathcal{C} .

To see that \mathcal{C} is not BC^* learnable, note that if some learner N BC^* learns \mathbb{N} , then there is a $\sigma \in (\mathbb{N} \cup \{\#\})^*$ such that for all $\tau \in (\mathbb{N} \cup \{\#\})^*$, $W_{N(\sigma\tau)}$ is cofinite: otherwise, one can build a text T' for \mathbb{N} such that N on T' outputs a coinfinite set infinitely often, contradicting the fact that N BC^* learns \mathbb{N} . If $\text{range}(\sigma) = \emptyset$, let $d = 0$; otherwise, let $d = \max(\text{range}(\sigma))$. Then one can extend σ to a text $\sigma \circ T''$ for $L' = \{0, \dots, d\} \cup \{2z : 2z > d\}$. By the choice of σ , N on $\sigma \circ T''$ almost always outputs a cofinite set, and so it does not even partially learn L' . Therefore \mathcal{C} is not BC^* learnable. ■

Remark 31. Note that *ApproxBC*Part* learning cannot in general be combined with consistency; for example, consider the class $\{K\}$, which is finitely learnable but cannot be consistently learnt because K is not recursive [8, Theorem 18].

While the preceding negative results suggest that approximate and weakly approximate learning imposes constraints that are too stringent for combining with partial learning, at least partly positive results can be obtained. For example, the following theorem shows that *ConsvPart* learnable classes are *ApproxPart* learnable (thus dropping only the conservativeness constraint) by BC^* learners. This considerably improves an earlier result by Gao, Stephan and Zilles [8] which states that every *ConsvPart* learnable class is also BC^* learnable.

Theorem 32. *If \mathcal{C} is ConsvPart learnable then \mathcal{C} is ApproxPart learnable by a BC^* learner.*

Proof. Let M be a *ConsvPart* learner for \mathcal{C} . For a text T for a language $L \in \mathcal{C}$, one considers the sequence e_0, e_1, \dots of distinct hypotheses issued by M ; it contains one correct hypothesis while all others are not indices of supersets of L . For each hypothesis e_n one has two numbers tracking its quality: $b_{n,t}$ is the maximal $s \leq n + t$ such that all $T(u)$ with $u < s$ are in $W_{e_n, n+t} \cup \{\#\}$ and $a_{n,t} = 1 + \max\{b_{m,t} : m < n\}$.

Now one defines the hypothesis set $H_{e_n, \sigma}$ for any sequence σ . Let $e_{n,0}, e_{n,1}, \dots$ be a sequence with $e_{n,0} = e_n$ and $e_{n,u}$ be the e_m for the minimum m such that

$m = n$ or W_{e_m} has enumerated all members of $\text{range}(\sigma)$ within $u + t$ time steps. The set $H_{e_n, \sigma}$ contains all x for which there is a $u \geq x$ with $x \in W_{e_n, u}$.

An intermediate learner O now conjectures some canonical index of a set $H_{e_n, \sigma}$ at least k times iff there is a t with $\sigma = T(0)T(1) \dots T(a_{n,t})$ and $b_{n,t} > k$. Thus O conjectures $H_{e_n, \sigma}$ infinitely often iff W_{e_n} contains $\text{range}(T)$ and $a_{n,t} = |\sigma|$ for almost all t .

If e_n is the correct index for the set to be learnt then, by conservativeness, the sets W_{e_m} with $m < n$ are not supersets of the target set. So the values $b_{m,t}$ converge which implies that $a_{n,t}$ converges to some s . It follows that for the prefix σ of T of length s , the canonical index of $H_{e_n, \sigma}$ is conjectured infinitely often while no other index is conjectured infinitely often. Thus O is a partial learner. Furthermore, for all sets $H_{e_m, \tau}$ conjectured after $a_{n,t}$ has reached its final value s , it holds that the $e_{m,u}$ in the construction of $H_{e_m, \tau}$ converge to e_n . Thus $H_{e_m, \tau}$ is the union of W_{e_n} and a finite set. Hence O is a BC^* learner. To guarantee the third condition on approximate learning, O will be translated into another learner N .

Let d_0, d_1, \dots be the sequence of O output on the text T . Now N will copy this sequence but with some delay. Assume that $N(\sigma_k) = d_k$ and σ_k is a prefix of T . Then N will keep the hypothesis d_k until the current prefix σ_{k+1} considered satisfies either $\text{range}(\sigma_{k+1}) \not\subseteq \text{range}(\sigma_k)$ or $W_{d_k, |\sigma_{k+1}|} \neq \text{range}(\sigma_{k+1})$.

If $\text{range}(T)$ is infinite, the sequence of hypotheses of N will be the same as that of O , only with some additional delay. Furthermore, almost all W_{d_n} contain $\text{range}(T)$, thus the resulting learner N learns $\text{range}(T)$ and is almost always correct on the infinite set $\text{range}(T)$; in addition, N learns $\text{range}(T)$ partially and is also BC^* . If $\text{range}(T)$ is finite, there will be some correct index that equals infinitely many d_n . There is a step t by which all elements of $\text{range}(T)$ have been seen in the text and enumerated into W_{d_n} . Therefore, when the learner conjectures this correct index again, it will never withdraw it; furthermore, it will replace eventually every incorrect conjecture due to the comparison of the two sets. Thus the learner converges explanatorily to $\text{range}(T)$ and is also in this case learning $\text{range}(T)$ in a BC^* way, partially and approximately. From the proof of Theorem 18, one can see that N may be translated into a learner satisfying all the requirements of *ApproxPart* and BC^* learning. ■

Example 33. The class $\{\{e+d : d \in \mathbb{N}\} : e \in \mathbb{N}\} \cup \{\{e+d : e \in K - K_d\} : e \in \mathbb{N}\}$ is *Ex* learnable and hence *ApproxBC*Part* learnable, but it is not *ConsvPart* learnable [6, Theorem 29].

Case and Smith [4] published Harrington's observation that the class of recursive functions is BC^* learnable. This result does not carry over to the class of r.e. sets; for example, Gold's class consisting of the set of natural numbers and all finite sets is not BC^* learnable. In light of Theorem 7, which established that the class of recursive functions can be BC^* and *Part* learnt simultaneously, it is interesting to know whether *any* BC^* learnable class of r.e. sets can be both BC^* and *Part* learnt at the same time. While this question in its general form

remains open, the next result shows that BC^n learning is indeed combinable with partial learning.

Theorem 34. *Let $n \in \mathbb{N}$. If \mathcal{C} is BC^n learnable, then \mathcal{C} is Part learnable by a BC^n learner.*

Proof. Fix any n such that \mathcal{C} is BC^n learnable. Given a recursive BC^n learner M of \mathcal{C} , one can construct a new learner N_1 as follows. First, let F_0, F_1, F_2, \dots be a one-one enumeration of all finite sets such that $|F_i| \leq n$ for all i . Fix a text T , and let e_0, e_1, e_2, \dots be the sequence of M 's conjectures on T .

For each set of the form $W_{e_i} \cup F_j$ (respectively $W_{e_i} - F_j$), N_1 outputs a canonical index for $W_{e_i} \cup F_j$ (respectively $W_{e_i} - F_j$) at least m times iff the following two conditions hold.

1. There is a stage $s > j$ for which the number of distinct $x < j$ such that either $x \in W_{e_i, s} \wedge x \notin \text{range}(T[s+1])$ or $x \in \text{range}(T[s+1]) \wedge x \notin W_{e_i, s}$ holds does not exceed n .
2. There is a stage $t > m$ such that for all $x < m$, $x \in W_{e_i, t} \cup F_j$ iff $x \in \text{range}(T[t+1])$ (respectively $x \in W_{e_i, t} - F_j$ iff $x \in \text{range}(T[t+1])$).

At any stage $T[s+1]$ where no set of the form $W_{e_i} \cup F_j$ or $W_{e_i} - F_j$ satisfies the conditions above, or each such set has already been output the required number of times (up to the present stage), N_1 outputs $M(T[s+1])$.

Suppose T is a text for some $L \in \mathcal{C}$. Since M is a BC^n learner of \mathcal{C} , it holds that for almost all i , there are at most n x 's such that $W_{e_i}(x) \neq L(x)$. Furthermore, for all j such that $W_{e_j}(x) \neq L(x)$ for at least $n+1$ distinct x 's, there is an l such that for all $l' > l$, neither $W_{e_j} \cup F_{l'}$ nor $W_{e_j} - F_{l'}$ will satisfy Condition 1.; thus, for any set S such that $S(x) \neq L(x)$ for more than n distinct values of x , N_1 will conjecture S only finitely often. On the other hand, if there are at most n distinct x 's such that $W_{e_i}(x) \neq L(x)$, then there is some l such that either $L = W_{e_i} \cup F_l$ or $L = W_{e_i} - F_l$; consequently, either $W_{e_i} \cup F_l$ or $W_{e_i} - F_l$ will satisfy Conditions 1. and 2. for infinitely many m . Hence N_1 is a BC^n learner of L and it outputs at least one correct index for L infinitely often on any text for L . Using a padding technique, one can define a further learner N that BC^n Part learns \mathcal{C} . ■

Theorems 35 and 38 show that partial BC^* learning is possible for classes that can be BC^* learned by learners that satisfy some additional constraints.

Theorem 35. *Assume that \mathcal{C} is BC^* learnable by a learner that outputs on each text for any $L \in \mathcal{C}$ at least once a fully correct hypothesis. Then \mathcal{C} is Part learnable by a BC^* learner.*

Proof. Let M be given and on a text T , let e_0, e_1, \dots be the sequence of hypotheses by M . Now one can make a learner O which on input $T(0)T(1) \dots T(n)$, first computes e_0, e_1, \dots, e_n and then computes for every e_m the quality $q_{m,n}$ which is the maximal number $y \leq n$ such that for all $x \leq y$ the number x has been enumerated into W_{e_m} iff $x \in \{T(0), T(1), \dots, T(n)\}$. In each step the learner O

outputs either the hypothesis for the least m such that either (a) e_m has been output so far less than $q_{m,n}$ times or (b) all $k \leq n$ satisfy that e_k has been output $q_{k,n}$ times and $q_{k,n} \leq q_{m,n}$. One can see that false hypotheses e_m get output only finitely often output while at least one correct hypotheses gets output infinitely often; as all but finitely many hypotheses of M are finite variants of L , the same is true for the modified learner O . By applying a padding technique, O can be converted to a learner N which is at the same time a BC^* learner and a partial learner. ■

The next definition gives an alternative way of tightening the constraint of BC^* learning.

Definition 36. Let \mathcal{C} be a class of r.e. sets. A recursive learner M is said to Vac^* learn \mathcal{C} iff M outputs on any text T for every $L \in \mathcal{C}$ only finitely many indices, and for almost all n , $W_{M(T[n+1])}$ is a finite variant of L .

Example 37. Case and Smith [4] showed that Vac^* and Ex^* learning of recursive functions are equivalent. However, this equivalence does not extend to all classes of r.e. sets. Take, for example, the class $\mathcal{C} = \{\{e\} \oplus \mathbb{N} : e \in \mathbb{N}\} \cup \{\{e\} \oplus \{x : x \leq |W_e|\} : e \in \mathbb{N}\}$. \mathcal{C} is Vac learnable: on any input σ whose range is of the form $\{e\} \oplus D$, determine whether $\max(D) > |W_{e,|\sigma|}|$; if so, conjecture $\{e\} \oplus \mathbb{N}$; otherwise, conjecture $\{e\} \oplus \{x : x \leq |W_e|\}$. If $range(\sigma)$ does not contain any even number, conjecture $range(\sigma)$.

On the other hand, \mathcal{C} is not Ex^* learnable. Assume by way of a contradiction that a recursive learner M Ex^* learns \mathcal{C} . Using K as an oracle, one can determine for any e whether W_e is finite. By the assumption that M is an Ex^* learner, one can enumerate a text T for $L_e = \{e\} \oplus \{x : x \leq |W_e|\}$ until at least one of the following holds.

1. There is some m such that for all $x > m$, $x \notin W_e$. This immediately implies that W_e is finite.
2. For some $\sigma \in (L_e \cup \{\#\})^*$ such that σ is a prefix of T , it holds that for all $\eta \in (L_e \cup \{\#\})^*$, $M(\sigma\eta) = M(\sigma)$; in other words, σ is a locking sequence for L_e .

Now one can use K again to determine whether or not there exists an $\eta \in (\{e\} \oplus \mathbb{N})^*$ such that $M(\sigma\eta) \neq M(\sigma)$. Suppose that $|W_e|$ is finite. Then $\{e\} \oplus \mathbb{N}$ is not a finite variant of L_e ; furthermore, as M must Ex^* learn $\{e\} \oplus \mathbb{N}$, there must exist some $\eta \in (\{e\} \oplus \mathbb{N})^*$ for which $M(\sigma\eta) \neq M(\sigma)$. Suppose, on the other hand, that $|W_e|$ is infinite. Then $L_e = \{e\} \oplus \mathbb{N}$, so that by the locking sequence property of σ , $M(\sigma\eta) = M(\sigma)$ for all $\eta \in (\{e\} \oplus \mathbb{N})^*$. Hence the Ex^* learnability of \mathcal{C} would imply that $\{e : |W_e| < \infty\}$ is Turing reducible to K , which is known to be false [20]. ■

Theorem 38. Suppose there is a recursive learner that BC^* learns \mathcal{C} and outputs on every text for any $L \in \mathcal{C}$ at least one index infinitely often. Then \mathcal{C} is BC^* Part learnable.

Proof. Let M be a recursive BC^* learner of \mathcal{C} such that M outputs on every text for any $L \in \mathcal{C}$ at least one index infinitely often. Define a learner N_1 as follows.

On any given text T for some $L \in \mathcal{C}$, let $e_n = M(T[n+1])$. Let F_0, F_1, F_2, \dots be a one-one enumeration of all finite sets. On input $T[k+1]$, N_1 outputs a canonical index $d_{e_k, l}$ for $W_{e_k} \cup F_l$ (respectively $g_{e_k, l}$ for $W_{e_k} - F_l$) at least m times iff the following conditions hold:

1. M outputs e_k at least $l+1$ times;
2. there is a stage $s > m$ such that for all $x < m$, $x \in \text{range}(T[s+1])$ iff $x \in W_{e_k, s} \cup F_l$ (respectively $x \in \text{range}(T[s+1])$ iff $x \in W_{e_k, s} - F_l$).

It will be shown that N_1 has the following two learning properties: first, it BC^* learns \mathcal{C} ; second, it outputs at least one correct index infinitely often; third, it outputs an incorrect index only finitely often. Consider any e_k .

First, suppose that W_{e_k} is not a finite variant of L . Then M outputs e_k only finitely often. Further, N_1 will consider sets of the form $W_{e_k} \cup F_l$ or $W_{e_k} - F_l$ for only finitely many F_l . Since, for each such $W_{e_k} \cup F_l$ (or $W_{e_k} - F_l$), item 2. will be satisfied for only finitely many m , it follows that N_1 will conjecture a set of the form $W_{e_k} \cup F_l$ or $W_{e_k} - F_l$ only finitely often.

Second, suppose that W_{e_k} is a finite variant of L . Then for any F_l , $W_{e_k} \cup F_l$ and $W_{e_k} - F_l$ are both finite variants of L . Hence N_1 preserves its BC^* learning property by outputting any indices for $W_{e_k} \cup F_l$ or $W_{e_k} - F_l$. Moreover, M outputs infinitely often at least one index e_h such that W_{e_h} is a finite variant of L . If $L = W_{e_h} \cup F_c$ (respectively $L = W_{e_h} - F_c$) for some F_c , then N will consider $W_{e_h} \cup F_c$ (respectively $W_{e_h} - F_c$) after M has output e_h at least $c+1$ times. As $W_{e_h} \cup F_c$ (respectively $W_{e_h} - F_c$) satisfies item 2. for almost all m , N_1 will output at least one index for L infinitely often.

Third, suppose that for some F_l , neither $W_{e_k} \cup F_l$ nor $W_{e_k} - F_l$ is equal to L . Then $W_{e_k} \cup F_l$ and $W_{e_k} - F_l$ will satisfy Condition 2. for all but finitely many m , and so N_1 will output a canonical index for $W_{e_k} \cup F_l$ or $W_{e_k} - F_l$ only finitely often. This establishes the three learning properties of N_1 .

Using a padding technique, one can define a further learner N such that N preserves the BC^* learning property of N_1 ; further, if e'_h is the minimum index that N_1 outputs infinitely often on T , then there is a h' with $e'_h = e_{h'}$ such that N will output $\text{pad}(e'_{h'}, d_{h'})$ infinitely often, and every other index is output only finitely often. Therefore N is both a BC^* and a *Part* learner of \mathcal{C} . ■

Corollary 39. *If a class \mathcal{C} of r.e. sets is Vac^* learnable, then \mathcal{C} is BC^* Part learnable.*

Example 40. Case and Smith [4] showed that the class of recursive functions $\mathcal{F} = \{f : f \text{ is recursive} \wedge \forall^\infty x [f = \varphi_{f(x)}]\}$ is BC learnable but not Ex^* learnable. By the equivalence of Ex^* and Vac^* in the setting of learning recursive functions, \mathcal{F} is also not Vac^* learnable. Furthermore, by Theorem 38, the class \mathcal{F} witnesses the separation of Vac^* and BC^* Part learnability.

The following proposition shows that two relatively strong learning criteria can be synthesized to produce quite a strict learning criterion.

Proposition 41. *If a class \mathcal{C} of r.e. sets is Vac^*WPart learnable, then \mathcal{C} is Vac learnable.*

Proof. Assume that M is a Vac^*WPart learner of \mathcal{C} . Define a new learner N as follows. On input σ , let e_0, e_1, \dots, e_k be all the distinct conjectures of M on prefixes of σ . For each e_i , let p_i be the maximum number such that for all $x < p_i$, $x \in W_{e_i, |\sigma|}$ holds iff x is contained in $range(\sigma)$. Furthermore, let $q = \max(\{p_i : 0 \leq i \leq k\})$ and m be the least index such that $p_m = q$; N then outputs e_m .

Let d_0, \dots, d_l be all the distinct conjectures of M on some text T for an $L \in \mathcal{C}$. Since M is a $WPart$ learner, it must output at least one index for L on T . Consider any d_i, d_j such that $W_{d_i} \neq L$ and $W_{d_j} = L$. Let z_i be the maximum number such that for all $x < z_i$, $x \in W_{d_i}$ holds iff $x \in L$. Then on almost all text prefixes $T[s]$, there must exist some $y_j > z_i$ such that for all $x < y_j$, $x \in W_{d_j, s+1}$ iff x is contained in $range(T[s])$. As there are only finitely many incorrect indices that M outputs, it follows that N will almost always output some index d_c for which $W_{d_c} = L$. Therefore N is a Vac learner of \mathcal{C} . ■

The following proposition implies that vacillatory learning cannot in general be combined with partial learning; in other words, a vacillatorily learnable class may not necessarily be vacillatorily as well as partially learnable at the same time.

Proposition 42. *If a class \mathcal{C} of r.e. sets is Vac^*Part learnable, then \mathcal{C} is Ex learnable.*

Proof. If M is a recursive learner of \mathcal{C} such that on any text T for some $L \in \mathcal{C}$, M outputs only finitely many indices and outputs exactly one index d for L infinitely often, then M almost always outputs d on T . ■

Example 43. The class of all cofinite sets is Ex^* learnable (and hence Vac^* learnable) but it is not Ex learnable. By Prop 42, this class is also not Vac^*Part learnable.

7 Conclusion

This paper studied conditions under which various forms of partial learning can be combined with models of approximation and with BC^* learning. For learning of recursive functions, it positively resolved Fulk and Jain's open question on whether the class of all recursive functions can be approximately learnt and BC^* learnt at the same time. For learning r.e. languages, three notions of approximate learning were introduced and studied. However, questions on the combinability of some pairs of learning constraints remain open. In particular, it is unknown whether or not every BC^* learnable class of r.e. languages has a learner that is both BC^* and $Part$.

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